From **NP**-completeness to **DP**-completeness: a general methodology to solve product families

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The class **DP**

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- $\mathbf{NP} \cup \mathbf{co} - \mathbf{NP} \subseteq \mathbf{DP} \subseteq \mathbf{PSPACE}$

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Main goals

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- Give a general methodology to solve products of problems.

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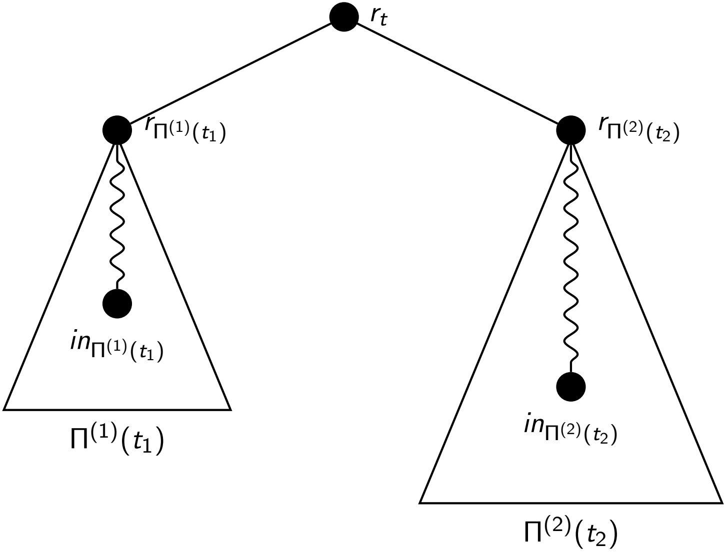
$\mathbf{\Pi^{(1)}} \otimes \mathbf{\Pi^{(2)}} \in \mathcal{R}'$ ($\mathcal{R} \subseteq \mathcal{R}'$) solves $X_1 \otimes X_2$ ($X_1 \otimes X_2$ is **DP**-complete).

General methodology (cell-like P systems)



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- **Output** of the computation.

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  - Objects do not have to reach r_t at the same time.

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- With **dissolution**, we can help objects to "**see**" where they are.

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- And... Which families are closed under product family?
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  - It is "easy" to get this behavior with some syntactic "ingredients".
- So... Ingredients are the key?
  - We have studied three ingredients: polarization, dissolution and minimal cooperation.

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Theorem

X is **NP***-complete, $X \in$* **PMC**$_{\mathcal{R}}$ *and $\mathcal{R}$ is closed under product family, then $X \otimes \overline{X} \in$* **PMC**$_{\mathcal{R}}$*, that is,* **DP** $\subseteq \mathcal{R}$.

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- so, $X \otimes \overline{X} \in$ **PMC**$_{\mathcal{R}}$ (by closure under product family).

□

THANKS FOR YOUR ATTENTION!